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# Diagnostic Map: Number

## Emergent Phase

### During the Emergent Phase

Students reason about small amounts of physical materials, learning to distinguish small collections by size and recognizing increases and decreases in them. They also learn to recognize and repeat the number words used in their communities and to distinguish number symbols from other symbols. There is a growing recognition of what is the same about the way students' communities use numbers to describe collections and what is different between collections labelled with different numbers.

**As a result, students come to understand that number words and symbols can be used to signify the "numerosity" of a collection.**

### By the end of the Emergent phase, students typically:

- use "bigger", "smaller" and "the same" to describe differences between small collections of like objects and between easily compared quantities
- anticipate whether an indicated change to a collection or quantity will make it bigger, smaller or leave it the same
- distinguish spoken numbers from other spoken words
- distinguish numerals from other written symbols
- see at a glance how many are in small collections and attach correct number names to such collections
- connect the differences they see between collections of one, two and three with the number string: "one, two, three, ..."
- understand a request to share in a social sense and distribute items or portions

**These students recognize that numbers may be used to signify quantity.**

## Matching Phase

Most students will enter the Matching phase between 3 and 5 years of age.

### As students move from the Emergent phase to the Matching phase, they:

- may actually see at a glance how many there are in a small collection, such as six pebbles, yet may not be able to say the number names in order
- may say a string of the number names in order (*one, two, three, four, ...*), but not connect them with how many are in collections
- may be beginning to see how to use the number names to count, but may get the order of the names wrong
- can tell by looking which of two small collections is bigger; however, they generally cannot say how much bigger
- may distribute items or portions in order to "share", but may not be concerned about whether everyone gets some, the portions are equal, or the whole amount is used up

### During the Matching Phase

Students use numbers as adjectives that describe actual quantities of physical materials. Through stories, games and everyday tasks, students use one-to-one relations to solve problems where they can directly carry out or imagine the actions suggested in the situation. They learn to fix small collections to make them match, "deal out" collections or portions, and to respect the principles of counting.

**As a result, students learn what people expect them to do in response to requests such as: How many are there? Can you give me six forks? How many are left? Give out one (two) each. Share them.**

### By the end of the Matching phase, students typically:

- recall the sequence of number names at least into double digits
- know how to count a collection, respecting most of the principles of counting
- understand that it is the last number said which gives the count
- understand that building two collections by matching one to one leads to collections of equal size, and can "fix" one collection to make it match another in size
- compare two collections one to one and use this to decide which is bigger and how much bigger
- solve small number story problems which require them to add some, take away some, or combine two amounts by imagining or role playing the situation and counting the resulting quantity
- share by dealing out an equal number of items or portions to each recipient, cycling around the group one at a time or handing out two or three at a time

**These students use one-to-one relations to share and count out.**

## Quantifying Phase

Most students will enter the Quantifying phase between 5 and 6 or more years of age.

### As students move from the Matching phase to the Quantifying phase, they:

- often do not spontaneously use counting to compare two groups in response to questions, such as: Are there enough cups for all students?
- may "skip count" but do not realize it gives the same answer as counting by ones and, therefore, do not trust it as a strategy to find how many
- often still think they could get a different answer if they started at a different place, so do not trust counting on or counting back
- often can only solve addition and subtraction problems when there is a specific action or relationship suggested in the problem situation which they can directly represent or imagine
- have difficulty linking their ideas about addition and subtraction to situations involving the comparison of collections
- may lay out groups to represent multiplicative situations, but do not use the groups to find out how many altogether, counting ones instead
- may represent division-type situations by sharing out or forming equal groups, but become confused about what to count to solve the problem, often choosing to count all the items
- may deal out an equal number of items or portions in order to share, but do not use up the whole quantity or attend to equality of the size of portions
- often do not realize that if they have shared a quantity, then counting one share will also tell them how many are in the other shares
- may split things into two portions and call them halves but associate the work "half" with the process of cutting or splitting and do not attend to equality of parts

### During the Quantifying Phase

Students reason about numerical quantities and come to believe that if nothing is added to, or removed from, a collection or quantity, then the total amount must remain the same even if its arrangement or appearance is altered.

**As a result, students see that the significance of the number uttered at the end of the counting process does not change with rearrangement of the collection or the counting strategy. They interpret small numbers as compositions of other numbers.**

**Also as a result, they develop the idea that constructing fair shares requires splitting the whole into equal parts without changing the total quantity and so begin to see the part-whole relations that link sharing and fractions.**

### By the end of the Quantifying phase, students typically:

- without prompting, select counting as a strategy to solve problems, such as: Are there enough cups? Who has more? Will it fit?
- use materials or visualize to decompose small numbers into parts empirically; *8 is the same as 5 with 3*
- find it obvious that when combining or joining collections counting on will give the same answer as starting at the beginning and counting the group
- make sense of the notion that there are basic facts, such as  $4 + 5$  is always 9, no matter how they work it out or in what arrangement
- select either counting on or counting back for subtraction problems, depending on which strategy best matches the situation
- can think of addition and subtraction situations in terms of the whole and the two parts and which is missing
- write number sentences that match how they think about the story line (semantic structure) for small number addition and subtraction problems
- realize that repeated addition or skip counting will give the same result as counting by ones
- realize that if they share a collection into a number of portions by dealing out or continuous halving and use up the whole quantity, then the portions must be equal regardless of how they look
- understand that the more portions to be made from a quantity, the smaller the size of each portion

**These students use part-part-whole relations for numerical quantities.**

## What Is the Diagnostic Map for Number?

As students' thinking about the key mathematical concepts of Number develops, it goes through a series of characteristic phases that are described in this Diagnostic Map. Recognizing these common patterns of thinking helps teachers to interpret students' responses to activities, to understand why students seem to be able to do some things and not others and also why some students may be having difficulty in achieving certain learning goals while others may not. The Diagnostic Map also helps teachers to provide the challenges students need to move their thinking forward, to refine any half-formed ideas, and to overcome any misconceptions students may have, so they can achieve the mathematical learning goals of Number.

See over



# Diagnostic Map: Number cont.

## Partitioning Phase

Most students will enter the Partitioning phase between 6 and 9 years of age.

As students move from the Quantifying phase to the Partitioning phase, they:

- often cannot decompose into parts numbers that they cannot visualize or represent as quantities, so have difficulty in partitioning larger numbers to make calculation easier; for example, students need to count forwards or backwards by ones to find the difference between 25 and 38
- often use strategies based on materials, counting on or counting back to solve addition and subtraction problems, but do not link these strategies or different problem types to a single operation (either + or -)
- may be unable to use the inverse relationship between addition and subtraction to choose the more efficient of counting on or counting back for solving particular problems
- often write their number sentences after they have solved the problem with materials, counting or basic facts, so they may be unable to write number sentences in advance when needed for problems involving larger numbers
- can count equal groups by physically or mentally laying out each group, but think of and treat each group as distinct from the others
- often believe that for two halves there must be exactly two pieces; for example, students may deny the equality of one-half and two-quarters unless the two-quarters are “stuck back together”
- although understanding that the two halves they have formed by dealing out or splitting must be equal, may think that a half formed one way could be bigger than a half formed another way
- may ignore the size of portions when choosing fraction names; for example, describing one part in seven as one-seventh regardless of whether the seven portions are equal
- often do not link sharing to unit fractions and may think that eighths are bigger than thirds because 8 is bigger than 3

### During the Partitioning Phase

Students come to see the significance of whole numbers having their own meaning independent of particular countable objects. They learn to use part-whole reasoning without needing to see or visualize physical collections.

**As a result, students see that numbers have magnitudes in relation to each other, can interpret any whole number as composed of two or more other numbers, and see the relationship between different types of addition and subtraction situations.**

**Also as a result, students see that numbers can be used to count groups and that they can use one group as a representative of other equal groups. They trust, too, that appropriate partitioning of quantities must produce equal portions.**

**By the end of the Partitioning phase, students typically:**

- can compare whole numbers using their knowledge of the patterns in the number sequence, and think of movements between numbers without actually or mentally representing the numbers as physical quantities
- make sense of why any whole number can be rewritten as the addition of other numbers
- partition at least two- and three-digit numbers into standard component parts ( $326 = 300 + 20 + 6$ ) without reference to actual quantities
- count up and down in tens from starting numbers like 23 or 79
- write suitable number sentences for the range of addition and subtraction situations
- use the inverse relationship between addition and subtraction to make a direct calculation possible; for example, re-interpret  $43 - 27$  as “what do you have to add to 27 to get 43” and so count on by tens and ones
- can double count in multiplicative situations by representing one group (*by holding up four fingers*) and counting repetitions of that same group, simultaneously keeping track of the number of groups and the number in each group
- find it obvious that two different-shaped halves from the same size whole must be the same size and are not tricked by perceptual features
- use successive splits to show that one-half is equivalent to 2 parts in 4, 4 parts in 8, and so on and expect that if the number of portions is doubled, they halve the size of each portion
- partition a quantity into a number of equal portions to show unit fractions and, given a particular quantity, will say that one-third is more than one-quarter

**These students use additive thinking to deal with many-to-one relations.**

## Factoring Phase

Most students will enter the Factoring phase between 9 and 11 years of age.

As students move from the Partitioning phase to the Factoring phase, they:

- can “work out” a non-standard partition ( $47 - 30 = 17$ ), but they may not see it as following automatically from the way numbers are written
- often do not realize that the digit in the tens (hundreds) place refers to groups of ten (hundred) even when they correctly use the labels “ones”, “tens” and “hundreds”
- have developed ideas about decimals based on daily use for money and measures, so may think the decimal point separates two whole numbers, where the whole numbers refer to different-sized units; for example, when referring to money, they may read 6.125 as if the 6 is dollars and the 125 is cents and thus “round” it to \$7.25 or say that  $6.125 > 6.25$
- may rightly think of decimals as another way to represent fractional numbers but, for example, think 0.6 is one-sixth
- often write related divisions and multiplications ( $6 \times 3 = 18$ ,  $18 \div 3 = 6$ ,  $18 \div 6 = 3$ ) by working each out, are unable to use the inverse relationship between division and multiplication to work out an unknown quantity
- may not understand why grouping can be used to solve a sharing problem
- can write multiplication number sentences for problems which they can think of as “groups of”, but may solve other types of multiplicative problems only with materials or by counting
- do not understand why multiplication is commutative; for example, they often do not see that four piles of 13 must be the same amount as 13 groups of 4
- may believe that to show a fraction of a collection the denominator must match the total number of items and will be unable, for example, to recognize six parts in 18 as one-third
- may think of  $\frac{1}{3}$  only as one part out of a collection or quantity which has been split into three equal parts, but do not also recognize it as one in each three
- may think of fractions as quantities rather than numbers and not see the significance of using the same unit as the basis for comparing fractions, so do not see why  $\frac{1}{3}$  must be bigger than  $\frac{1}{4}$
- may see fractions, such as three-quarters, literally as three pieces each of one-quarter and will not accept one piece which is three-quarters of the whole

### During the Factoring Phase

Students extend their additive ideas about whole numbers to include the coordination of two factors needed for multiplicative thinking. They learn to construct and coordinate groups of equal size, numbers of groups and a total amount. Students also learn to visualize multiplicative situations in terms of a quantity arranged in rows and columns (an array).

**As a result, students see the significance of the connection between groups of ten or groups of one hundred and the way we write whole numbers. They are able to relate different types of multiplication and division situations involving whole numbers. They also link the ideas of repeating equal groups, splitting a quantity into equal parts and fractions.**

**By the end of the Factoring phase, students typically:**

- use their knowledge to generate alternative partitions, for example, the 2 in the tens place in 426 refers to 2 groups of 10
- sustain a correct whole number place-value interpretation in the face of conflicting information
- are flexible in their mental partitioning of whole numbers, confident that the quantity has not changed
- understand that a number can be decomposed and re-composed into its factors in a number of ways without changing the total quantity
- find it obvious that if 3 rows of 5 is 15, then both 15 divided by 3 and one-third of 15 are 5
- can visualize an array to see, for example, that five blue counters is one-third of a bag of 15 counters, both because 15 can be split into three parts each of five and one in every three counters will be blue
- visualize or draw their own diagrams to compare fractions with the same denominator ( $\frac{3}{7}$  and  $\frac{5}{7}$ ) or simple equivalences ( $\frac{1}{4}$  and  $\frac{2}{8}$ )
- use the idea of splitting a whole into parts to understand, for example, that 2.4 is  $2 + \frac{4}{10}$  and 2.45 is  $2 + \frac{45}{100}$
- relate fractions and division knowing, for example, that  $\frac{3}{4}$  can be thought of as  $3 \div 4$  and 3 things shared among 4 students has to be  $\frac{3}{4}$
- know that they can choose between multiplication or division to make calculating easier
- understand why grouping and sharing problems can be solved by the same division process
- interpret multiplication situations as “times as much” and so can see that 12 is 3 times as much as 4, and 8 is 10 times smaller than 80
- select an appropriate multiplication or division operation on whole numbers including for problems that are not easily interpreted as “groups of”; for example, combination and comparison problems
- can see why multiplication of whole numbers is commutative; for example, knowing without calculating, that 4 piles of 9 objects must be the same amount as 9 piles of 4 objects

**These students think both additively and multiplicatively about numerical quantities.**

## Operating Phase

Most students will enter the Operating phase between 11 and 13 years of age.

As students move from the Factoring phase to the Operating phase, they:

- often continue to rely largely on their knowledge of the “named” places in reading and writing numbers, so have difficulty writing numbers with more than four digits
- may label the places to the right of the decimal point as tenths and hundredths and write 2.45 as  $2 + \frac{4}{10} + \frac{5}{100}$ , for example, but cannot link this with other ways of writing the decimal, such as  $2 + \frac{45}{100}$
- may think decimals with two places are always hundredths and write 2.45 as  $2 + \frac{45}{100}$ , but do not link this with the pattern in whole-number place value and so do not see 2.45 as  $2 + \frac{4}{10} + \frac{5}{100}$
- often are unable to select a common partitioning (denominator) to enable two fractions to be compared or combined unless an equivalence they already know is involved
- often ignore the need to draw two fractions on identical wholes in order to compare or combine them
- may be unable to select an appropriate operation in situations where they cannot think of the multiplier or divisor as a whole number
- may resist selecting division where the required division involves dividing a number by a bigger number
- often believe that multiplication “makes bigger” and division “makes smaller”

### During the Operating Phase

Students learn to interpret multipliers as “times as much as” or “of” rather than simply counters of groups, so can think of them as “operators” that need not be whole numbers. Students also come to see that any number can be thought of as a unit which can be repeated or split up any number of times.

**As a result, students see how the intervals between whole numbers can be split and re-split into increasingly smaller intervals and realize the significance of the relationship between successive places. For example, the value of each place is ten times the value of the place to its right and one-tenth of the value of the place to its left.**

**Also as a result, students learn to make multiplicative comparisons between numbers, deal with proportional situations, and integrate their ideas about common and decimal fractions.**

**By the end of the Operating phase, students typically:**

- represent common and decimal fractions both smaller and greater than 1 on a number line
- generalize their understanding of whole-number place value to include the cyclical pattern beyond the thousands, so can read, write and say any whole numbers
- use their understanding of the relationship between successive places to order decimal numbers regardless of the number of places
- use the cyclical pattern in the places to count forwards and backwards in tenths, hundredths, thousandths, including up and over whole numbers
- are flexible in partitioning decimal numbers
- realize that for multipliers smaller than 1, multiplication makes smaller, and for divisors smaller than 1, division makes bigger
- select an appropriate number of partitions to enable a quantity to be shared into two different numbers of portions; such as 5 or 3
- construct successive partitions to model multiplication situations; *I took half the cake home and then ate one-third of it*
- produce their own diagrams to compare or combine two fractions, ensuring that both fractions ( $\frac{2}{3}$  and  $\frac{1}{4}$ ) are represented on identical wholes
- split and recombine fractions visually or mentally to add or subtract;  $\frac{1}{2} + \frac{1}{4}$  is  $(\frac{1}{4} + \frac{1}{4}) + \frac{1}{4} = \frac{3}{4}$
- recognize the need to multiply in situations where the multiplier is a fractional number
- can write suitable number sentences for the full range of multiplication and division situations involving whole numbers, decimals and fractions

**These students can think of multiplications and divisions in terms of operators.**